Univerzitet u Zenici
Pedagoški fakultet
Odsjek: Matematika i informatika
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Pismeni ispit iz predmeta Linearna algebra

Bitna napomena: Svaku formulu koju mislite koristit, u sva 4 zadatka, obavezno napisati, kao i značenja simbola iz formule. Ispit pisati isključivo hemiskom olovkom plave ili crne tinte. Prije rješenja prepisati postavku (tekst) zadatka.

1. Neka je $\mathcal{L} = \{(x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_1 + x_2 = 0,\ x_1 + x_2 + x_3 = 0,\ x_1 + x_2 + x_3 + x_4 = 0\}$. Dokazati da je $\mathcal{L}$ podprostor od $\mathbb{R}^4$, odrediti mu bazu, dimenziju i neki direktni komplement (koji nije ortogonalni komplement).

2. (20%)(*a*) Neka je

$$T : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$

linearni operator definisan sa

$$T(x, y, z) = (x - y, z, y - x).$$

Odrediti jezgru i sliku od T i izračunati njihove dimenzije.

(80%)(*b*) Na vektorskom prostoru $\mathbb{R}_2[x]$ (polinoma sa realnim koeficijentima stepena najviše 2) zadan je linearni operator sa

$$D_\alpha(p(x)) = \frac{1}{\alpha}(p(x + \alpha) - p(x)), \quad \alpha \in \mathbb{R},\ \alpha \neq 0.$$

Odrediti jezgru i sliku od D_α.

3. Neka je T linearni operator na prostoru $\mathbb{R}^2$ koji djeluje tako da vektor prvo zarotira za ugao $\frac{\pi}{2}$ u pozitivnom sjeru, pa dobijeni vektor zatim reflektuje u odnosu na x-osu.

(*a*) Odrediti matricu operatora T (tj., matricu koordinata od T) u bazi $\mathcal{B} = \left\{ \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$.

(*b*) Odrediti $v \in \mathbb{R}^2$ takav da $T(v) = v$ i $\|v\| = 2\sqrt{2}$.

4. Dat je unitarni prostor $\mathcal{P}_3$, polinoma stepena ≤ 3, sa skalarnim (unutrašnjim) proizvodom

$$\langle p, q \rangle = \int_0^1 p(x)q(x)\ \mathrm{d}x.$$

Neka je $\mathcal{L}$ podprostor unitarnog prostora $\mathcal{P}_3$, generisan vektorima $p_1(x) = 1 - x$ i $p_2(x) = x^2 - x$. Odrediti neku bazu za ortogonalni kompliment od $\mathcal{L}$. Nadalje, prikažite $p(x) = 1 - 2x + 5x^3$ u obliku $q(x) + r(x)$, gdje je $q \in \mathcal{L}$, a $r \in \mathcal{L}^\perp$.

(#) Neka je $\mathcal{L}=\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid x_1+x_2=0,\ x_1+x_2+x_3=0,\ x_1+x_2+x_3+x_4=0\}$.

Dokazati da je $\mathcal{L}$ podprostor od $\mathbb{R}^4$, odrediti mu bazu, dimenziju i neki direktni komplement (koji nije ortogonalni komplement).

Rj.

$$\mathcal{L}=\{(x_1,x_2,x_3,x_4)\in\mathbb{R}^4 \mid x_1+x_2=0,\ x_1+x_2+x_3=0,\ x_1+x_2+x_3+x_4=0\}$$

$$=\left\{\begin{pmatrix}x_1\\x_2\\x_3\\x_4\end{pmatrix}\in\mathbb{R}^4 \;\middle|\; \begin{bmatrix}1&1&0&0\\1&1&1&0\\1&1&1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix}=0\right\}=$$

$$=\left\{\mathbf{x}\in\mathbb{R}^4 \;\middle|\; A\mathbf{x}=0 \text{ gdje je } A=\begin{pmatrix}1&1&0&0\\1&1&1&0\\1&1&1&1\end{pmatrix}\right\}$$

$$=\ker(A)$$

Prisjetimo se

Podprostor

Neka je $\mathcal{S}$ neprazan podskup vektorskog prostora $\mathcal{V}$ nad $\mathbb{F}$. Ako je $\mathcal{S}$ također vektorski prostor nad $\mathbb{F}$ sa istim operacijama sabiranja i skalarnog množenja, tada za $\mathcal{S}$ kažemo da je podprostor od $\mathcal{V}$. Nije neophodno provjeriti svih 10 uslova iz definicije da bi odredili da li je podskup također i podprostor - treba provjeriti samo uslove zatvorenosti (A1) i (M1). Tj. neprazan podskup $\mathcal{S}$ vektorskog prostora $\mathcal{V}$ je podprostor od $\mathcal{V}$ akko

(A1) $x,y\in\mathcal{S} \Rightarrow x+y\in\mathcal{S}$

(M1) $x\in\mathcal{S} \Rightarrow \alpha x\in\mathcal{S}$ za $\forall\alpha\in\mathbb{F}$

Prvo primijetimo da je $\mathcal{L}$ neprazan npr. $\begin{pmatrix}1\\-1\\0\\0\end{pmatrix}\in\mathcal{L}$, (A1) Pokažimo da vrijedi $x, y \in \mathcal{L} \Rightarrow x+y \in \mathcal{L}$

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathcal{L}$ i $y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix} \in \mathcal{L} \Rightarrow Ax = 0$ i $Bx = 0$

a kako je $A(x+y) = Ax + bx = 0$ to je $x+y \in \mathcal{L}$

vrijedi (A1)

(M1) Pokažimo da vrijedi $x \in \mathcal{L} \Rightarrow \alpha x \in \mathcal{L}$ za $\forall \alpha \in \mathbb{R}$.

$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \in \mathcal{L} \Rightarrow Ax = 0$

$$\Rightarrow A\alpha x = \alpha A x = \alpha \cdot 0 = 0 \Rightarrow \alpha x \in \mathcal{L}$$

vrijedi (M1)

$\mathcal{L}$ je podprostor vektorskog prostora $\mathbb{R}^4$.

Baza za $\mathcal{L} = \ker(A)$ je opšte rješenje sistema $Ax = 0$.

$$A = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{pmatrix} \underset{II_v - I_v}{\overset{III_v - II_v}{\sim}} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \Rightarrow \begin{matrix} \ker(A) = 3 \\ \ker(\bar{A}) = 3 \end{matrix}$$

broj nepoznatih $= 4$

$\Rightarrow$ jednu promjenjivu uzimamo proizvoljno

$$\begin{aligned} x_1 + x_2 \quad &= 0 \\ x_3 &= 0 \\ x_4 &= 0 \end{aligned}$$

$x_2 = t \Rightarrow x_1 = -t$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} -t \\ t \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} t \Rightarrow \mathcal{L} = \mathrm{span}\left\{ \begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \right\} \Rightarrow \dim \mathcal{L} = 1$$

Baza za $\mathcal{L}$ je $\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$, a dimenzija od $\mathcal{L}$ je 1.

Odredimo još direktni komplement

Komplementarni podprostor

Za podprostore $\mathcal{X}, \mathcal{Y}$ prostora $\mathcal{V}$ kažemo da su komplementarni akko $\mathcal{V}=\mathcal{X}+\mathcal{Y}$ i $\mathcal{X}\cap\mathcal{Y}=0$ i u tom slučaju kažemo da je $\mathcal{V}$ direktna suma od $\mathcal{X}$ i $\mathcal{Y}$, i ovo označavamo sa $\mathcal{V}=\mathcal{X}+\mathcal{Y}$.

$$\mathcal{X}+\mathcal{Y} := \{x+y \mid x\in\mathcal{X} \text{ i } y\in\mathcal{Y}\}.$$

Ako $\mathcal{X}$ i $\mathcal{Y}$ imaju redom baze $\mathcal{B}_{\mathcal{X}}$ i $\mathcal{B}_{\mathcal{Y}}$ tada vrijedi sljedeće

$$\mathcal{V}=\mathcal{X}\oplus\mathcal{Y} \iff \forall v\in\mathcal{V}\ \exists!\, x\in\mathcal{X}, y\in\mathcal{Y} \quad v=x+y \iff \mathcal{B}_{\mathcal{X}}\cup\mathcal{B}_{\mathcal{Y}}=\emptyset$$

$\mathcal{B}_{\mathcal{X}}\cup\mathcal{B}_{\mathcal{Y}}$ je baza za $\mathcal{V}$

Da bismo odredili direktni komplement za $\mathcal{L}$ nadopunimo bazu za $\mathcal{L}$ do baze za vektorski prostor $\mathbb{R}^4$.

$$\begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \overset{II_v + I_v}{\sim} \begin{pmatrix} -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Direktni komplement od $\mathcal{L}$ je $\operatorname{span}\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right\}$.

Neka je $T:\mathbb{R}^3 \to \mathbb{R}^3$ linearni operator definisan sa

$$T(x,y,z)=(x-y,\ z,\ y-x).$$

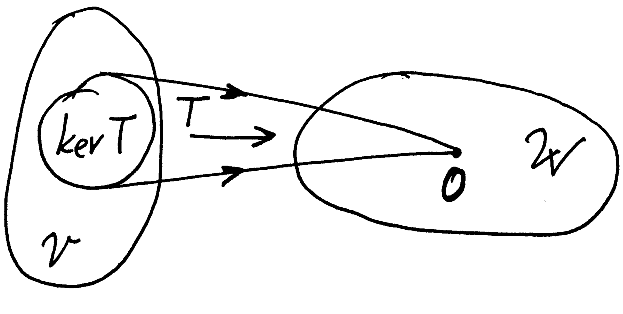
Odrediti jezgru i sliku od T i izračunati njihove dimenzije.

Rj.

$$T\begin{pmatrix}a\\b\\c\end{pmatrix}=\begin{pmatrix}a-b\\c\\b-a\end{pmatrix}$$

$$\ker T=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \,\middle|\, T\begin{pmatrix}x\\y\\z\end{pmatrix}=\mathbf{0}\right\}=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \,\middle|\, T\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}\right\}=$$

$T: V \to W$



$$=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \,\middle|\, x-y=0,\ z=0,\ y-x=0\right\}$$

$$=\left\{\begin{pmatrix}x\\y\\z\end{pmatrix}\in\mathbb{R}^3 \,\middle|\, \underbrace{\begin{bmatrix}1&-1&0\\0&0&1\\-1&1&0\end{bmatrix}}_{=A}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}\right\}$$

$$=\ker(A)$$

Jezgro od matrice A formiramo na osnovu opšteg rješenja sistema $\begin{bmatrix}1&-1&0\\0&0&1\\-1&1&0\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

$$A=\begin{bmatrix}1&-1&0\\0&0&1\\-1&1&0\end{bmatrix}\overset{III_v+I_v}{\sim}\begin{bmatrix}1&-1&0\\0&0&1\\0&0&0\end{bmatrix}\Rightarrow \begin{matrix}x-y=0\\z=0\end{matrix}\quad \Rightarrow x=y$$

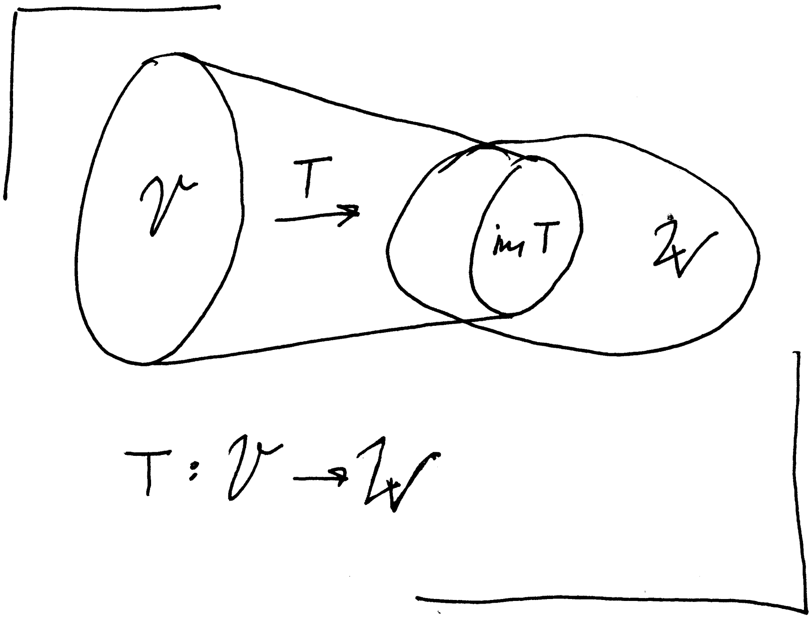
$y=t \Rightarrow x=t$

$\operatorname{rang}(A)=\operatorname{rang}(\bar{A})=2<3$

jednu promjenjivu uzimamo proizvoljno

$$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}t\\t\\0\end{pmatrix}=\begin{pmatrix}1\\1\\0\end{pmatrix}t$$

$\ker T=\operatorname{span}\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$ iz čega vidimo da je baza za $\ker T$ $\left\{\begin{pmatrix}1\\1\\0\end{pmatrix}\right\}$, pa je $\dim(\ker(T))=1$.



$T: V \to W$

$$\mathrm{im}(T) = \left\{ T\begin{pmatrix} x \\ y \\ z \end{pmatrix} \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\}$$

$$= \left\{ \begin{pmatrix} x-y \\ z \\ y-x \end{pmatrix} \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\}$$

$$= \left\{ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} \;\middle|\; \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \right\}$$

$$= \{ A\mathbf{x} \mid \mathbf{x} \in \mathbb{R} \} =$$

$$= \mathrm{im}(A)$$

im(A) je prostor generisan pomoću kolona matrice A.

Kako je $A \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$

$$\Rightarrow \quad \mathrm{im}(A) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

tj. $\mathrm{im}(T) = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$\dim(\mathrm{im}(T)) = 2$$

(#) Na vektorskom prostoru $\mathbb{R}_2[x]$ (polinoma sa realnim koeficijentima stepena najviše 2) zadan je linearni operator sa

$$D_\alpha(p(x)) = \frac{1}{\alpha}(p(x+\alpha)-p(x)), \quad \alpha\in\mathbb{R}, \alpha\neq 0.$$

Odredite jezgru i sliku od D_α.

Rj: Primjetimo da

$$\begin{aligned} D_\alpha : \mathbb{R}_2[x] &\longrightarrow \mathbb{R}_2[x] \\ p(x) &\longrightarrow \frac{1}{\alpha}(p(x+\alpha)-p(x)) \end{aligned}$$

Neka je $p(x)=ax^2+bx+c$. Izračunajmo $D_\alpha(p(x))$.

$$D_\alpha(p(x)) = \frac{1}{\alpha}(p(x+\alpha)-p(x)) = \frac{1}{\alpha}(a\overbrace{(x+\alpha)^2}^{x^2+2\alpha x+\alpha^2}+b(x+\alpha)+c-ax^2-bx-c)$$

$$= \frac{1}{\alpha}(2a\alpha x+a\alpha^2+b\alpha) = 2ax+a\alpha+b$$

Standardna baza za $\mathbb{R}_2[x]$ je $\varphi=\{1,x,x^2\}$ pa imamo

$$[p]_\varphi = \begin{pmatrix} c\\ b\\ a\end{pmatrix} \qquad [D_\alpha(p)]_\varphi = \begin{pmatrix} a\alpha+b\\ 2a\\ 0\end{pmatrix}$$

Prisjetimo se

<u>Neka je $T\in\mathcal{L}(\mathcal{U},\mathcal{V})$ i neka su $\mathcal{B}$ i $\mathcal{B}'$ redom baze za $\mathcal{U}$ i $\mathcal{V}$. Tada za $\forall u\in\mathcal{U}$ imamo $[T(u)]_{\mathcal{B}'}=[T]_{\mathcal{B}\mathcal{B}'}[u]_{\mathcal{B}}$.</u>

Odredimo $[D_\alpha]_\varphi$.

$$[D_\alpha]_\varphi = \begin{pmatrix} | & | & | \\ [D_\alpha(1)]_\varphi & [D_\alpha(x)]_\varphi & [D_\alpha(x^2)]_\varphi \\ | & | & | \end{pmatrix}$$

$$D_\alpha(1) = \frac{1}{\alpha}(1-1) = 0 \quad \Rightarrow \quad [D_\alpha(1)]_\varphi = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$D_\alpha(x) = \frac{1}{\alpha}(x+\alpha-x) = 1 \quad \Rightarrow \quad [D_\alpha(x)]_\varphi = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

$$D_\alpha(x^2) = \frac{1}{\alpha}\left((x+\alpha)^2 - x^2\right) = \frac{1}{\alpha}(2x\alpha+\alpha^2) = 2x+\alpha \quad \Rightarrow \quad [D_\alpha(x^2)]_\varphi = \begin{pmatrix} \alpha \\ 2 \\ 0 \end{pmatrix}$$

Prema tome

$$[D_\alpha]_\varphi = \begin{pmatrix} 0 & 1 & \alpha \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

Sad nije teško izračunati jezgru i sliku od D_α

$$\ker(D_\alpha) = \{ p(x) \in \mathbb{R}_2[x] \mid D_\alpha(p(x)) = 0 \} =$$

$$= \{ [p]_\varphi \mid [D_\alpha]_\varphi [p]_\varphi = 0 \}$$

tj. tražimo nepoznate a, b, c t.d. $\underbrace{\begin{pmatrix} 0 & 1 & \alpha \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}}_{=A} \begin{pmatrix} c \\ b \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$\left.\begin{matrix} \text{rang}(A) = 2 \\ \text{rang}(\bar{A}) = 3 \end{matrix}\right\}$ 1 promjenjivu uzimamo proizvoljno

$c = t, \ a = 0, \ b = 0$

$$\begin{aligned} a + \alpha b &= 0 \\ 2a &= 0 \\ \hline a = 0 \quad & b = 0 \end{aligned}$$

Prema tome $\ker(D_\alpha) = \mathbb{R}_0[x] = \{ t \mid t \in \mathbb{R} \}$.

$$\text{im}(D_\alpha) = \{ D_\alpha(p(x)) \mid p(x) \in \mathbb{R}_2[x] \} = \{ [D_\alpha]_\varphi [p]_\varphi \mid p \in \mathbb{R}_2[x] \}$$

$$= \text{im}\, A = \text{span}\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \alpha \\ 2 \\ 0 \end{pmatrix} \right\} = \mathbb{R}_1[x] = \{ \gamma x + \delta \mid \gamma, \delta \in \mathbb{R} \}.$$

(#) Neka je T linearni operator na $\mathbb{R}^2$ koji djeluje tako da vektor prvo zarotira za ugao $\frac{\pi}{2}$ u pozitivnom smjeru, pa dobijeni vektor zatim reflektuje u odnosu na x-osu.

(a) Odrediti matricu operatora T (tj. matricu koordinata od T) u bazi $\mathcal{B}=\left\{\begin{pmatrix}1\\1\end{pmatrix},\begin{pmatrix}2\\1\end{pmatrix}\right\}$.

(b) Odrediti $v\in\mathbb{R}^2$ takav da $T(v)=v$ i $\|v\|=2\sqrt{2}$.

Rj: Matrica koordinata

Neka su $\mathcal{B}=\{u_1,u_2,\dots,u_n\}$ i $\mathcal{B}'=\{v_1,v_2,\dots,v_m\}$ redom baze za $\mathcal{U}$ i $\mathcal{V}$. Matrica koordinata od $T\in\mathcal{L}(\mathcal{U},\mathcal{V})$ u odnosu na par $(\mathcal{B},\mathcal{B}')$ je definisana kao $m\times n$ matrica

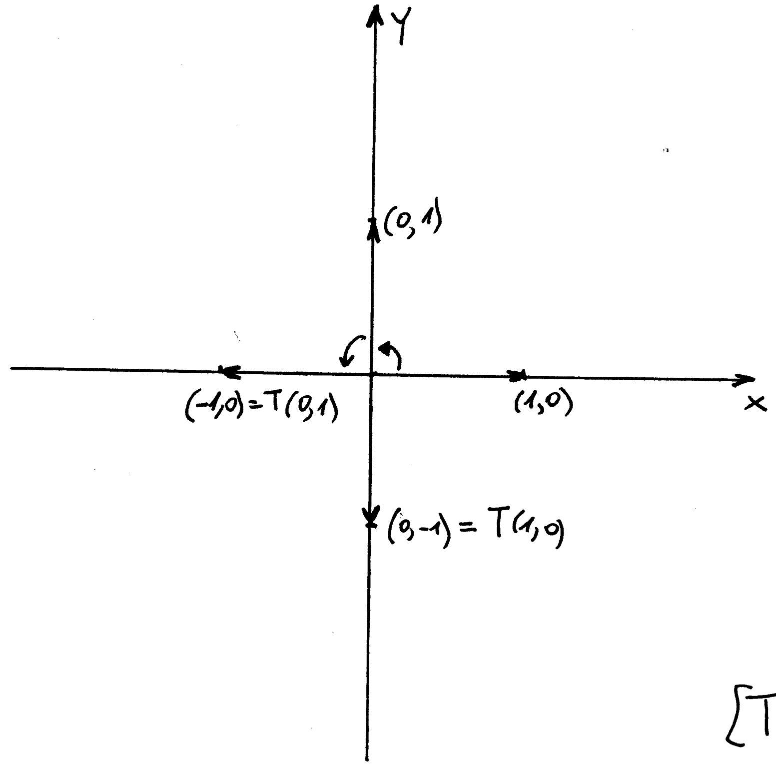
$$[T]_{\mathcal{B}\mathcal{B}'}=\begin{pmatrix} | & | & & | \\ [T(u_1)]_{\mathcal{B}'} & [T(u_2)]_{\mathcal{B}'} & \dots & [T(u_n)]_{\mathcal{B}'} \\ | & | & & | \end{pmatrix}$$

Kada je T linearni operator $\mathcal{U}$, i ako je u igri samo jedna baza, koristimo $[T]_{\mathcal{B}}$ umjesto $[T]_{\mathcal{B}\mathcal{B}}$.

Sa $\mathcal{S}$ označimo standardnu bazu $\mathcal{S}=\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ te odredimo matricu operatora T u bazi $\mathcal{S}$, te u bazi $\mathcal{B}$.

$$[T]_{\mathcal{S}}=\begin{pmatrix} | & | \\ \left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_{\mathcal{S}} & \left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_{\mathcal{S}} \\ | & | \end{pmatrix},\qquad [T]_{\mathcal{B}}=\begin{pmatrix} | & | \\ \left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_{\mathcal{B}} & \left[T\begin{pmatrix}2\\1\end{pmatrix}\right]_{\mathcal{B}} \\ | & | \end{pmatrix}$$

Odredimo $T\begin{pmatrix}1\\0\end{pmatrix}$ te $T\begin{pmatrix}0\\1\end{pmatrix}$.



Sa slike vidimo da

$$T\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\-1\end{pmatrix}$$

$$T\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}-1\\0\end{pmatrix}$$

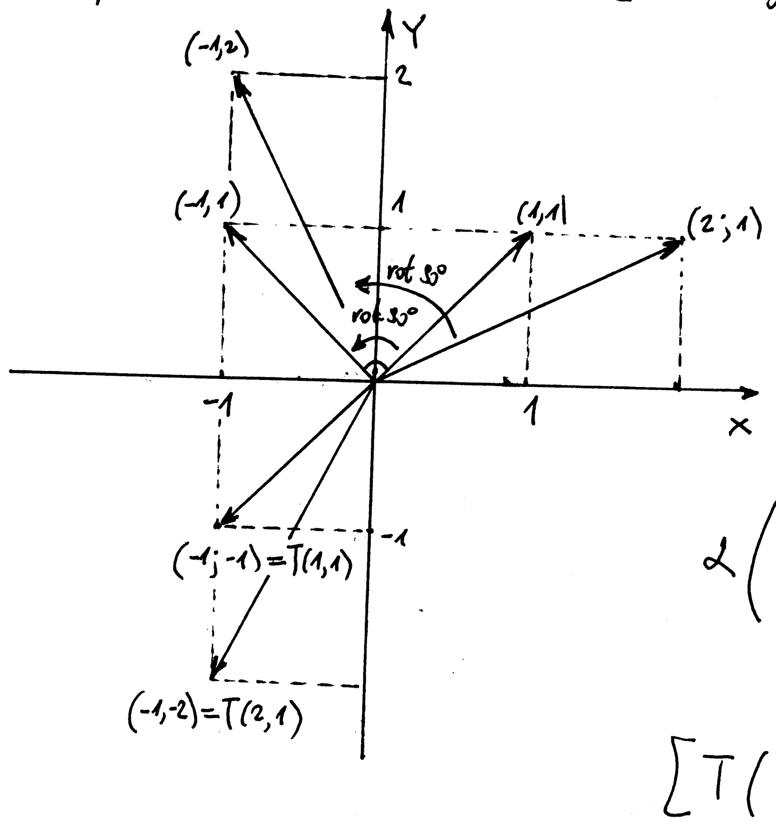
$$\Rightarrow \left[T\begin{pmatrix}1\\0\end{pmatrix}\right]_\varphi = \begin{pmatrix}0\\-1\end{pmatrix}$$

$$\left[T\begin{pmatrix}0\\1\end{pmatrix}\right]_\varphi = \begin{pmatrix}-1\\0\end{pmatrix}$$

$$[T]_\varphi = \begin{pmatrix}0 & -1\\-1 & 0\end{pmatrix}$$

a)

Pokušajmo sad odrediti $\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_\mathcal{B}$ i $\left[T\begin{pmatrix}2\\1\end{pmatrix}\right]_\mathcal{B}$.



Sa slike vidimo da

$$T\begin{pmatrix}1\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix}$$

$$T\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\alpha\begin{pmatrix}1\\1\end{pmatrix} + \beta\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\end{pmatrix} \Rightarrow$$

$$\alpha = -1, \quad \beta = 0$$

$$\left[T\begin{pmatrix}1\\1\end{pmatrix}\right]_\mathcal{B} = \begin{pmatrix}-1\\0\end{pmatrix}$$

$$\gamma\begin{pmatrix}1\\1\end{pmatrix} + \delta\begin{pmatrix}2\\1\end{pmatrix} = \begin{pmatrix}-1\\-2\end{pmatrix}$$

$$\begin{aligned} \gamma + 2\delta &= -1\\ -\quad \gamma + \delta &= -2\\ \hline \delta &= 1 \end{aligned}$$

$$\gamma = -2-1$$
$$\gamma = -3$$

$$\Rightarrow \left[T\begin{pmatrix}2\\1\end{pmatrix}\right]_\mathcal{B} = \begin{pmatrix}-3\\1\end{pmatrix}$$

$$[T]_\mathcal{B} = \begin{pmatrix}-1 & -3\\0 & 1\end{pmatrix}$$

tražena matrica operatora T u bazi $\mathcal{B} = \left\{\begin{pmatrix}1\\1\end{pmatrix}, \begin{pmatrix}2\\1\end{pmatrix}\right\}$

b) Odredimo vektor $v \in \mathbb{R}^2$ takav da $T(v) = v$ i $\|v\| = 2\sqrt{2}$.

Prisjetimo se

Neka je $T \in \mathcal{L}(\mathcal{U}, \mathcal{V})$ i neka su $\mathcal{B}$ i $\mathcal{B}'$ redom baze za $\mathcal{U}$ i $\mathcal{V}$. Tada za $\forall u \in \mathcal{U}$ imamo $[T(u)]_{\mathcal{B}'} = [T]_{\mathcal{B}\mathcal{B}'} [u]_{\mathcal{B}}$.

Pa ako uzmemo bazu $\varphi = \left\{ \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \end{pmatrix} \right\}$ imamo

$$[T(v)]_\varphi = [T]_\varphi \cdot [v]_\varphi$$

$$v = \begin{pmatrix} x \\ y \end{pmatrix} \Rightarrow [v]_\varphi = \begin{pmatrix} x \\ y \end{pmatrix}$$

$$T(v) = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ -x \end{pmatrix}$$

Kako trebamo imati $T(v) = v$ to je $\begin{matrix} x = -y \\ y = -x \end{matrix}$ tj. $x + y = 0$

Dalje, kako je $\|v\| = 2\sqrt{2}$ to je $\|v\|^2 = 8$

$$x^2 + y^2 = 8$$

$$\begin{aligned} x + y &= 0 \Rightarrow y = -x \\ x^2 + y^2 &= 8 \end{aligned}$$

$$\begin{aligned} x^2 + x^2 &= 8 \\ 2x^2 &= 8 \\ x^2 &= 4 \\ x_{1,2} &= \pm 2 \end{aligned}$$

Vektori v koji zadovoljavaju dati uslov su $\begin{pmatrix} 2 \\ -2 \end{pmatrix}, \begin{pmatrix} -2 \\ 2 \end{pmatrix}$.

(#) Dat je unitarni prostor $\mathcal{P}_3$, polinoma stepena ≤ 3, sa skalarnim (unutrašnjim) proizvodom

$$\langle p, q \rangle = \int_0^1 p(x)\, q(x)\, dx .$$

Neka je $\mathcal{L}$ podprostor unitarnog prostora $\mathcal{P}_3$, razapet (generisan) vektorima $p_1(x) = 1-x$ i $p_2(x) = x^2 - x$. Odrediti neku bazu za ortogonalni komplement od $\mathcal{L}$. Nadalje, prikažite $p(x) = 1 - 2x + 5x^3$ u obliku $q(x) + r(x)$, gdje je $q \in \mathcal{L}$, a $r \in \mathcal{L}^{\perp}$.

Rj. $\mathcal{L} = \operatorname{span}\{1-x,\ x^2 - x\}$

Ortogonalni komplement

Za skup $\mathcal{M}$ unitarnog prostora $\mathcal{V}$, ortogonalni komplement $\mathcal{M}^{\perp}$ od $\mathcal{M}$ je definisan sa

$$\mathcal{M}^{\perp} = \{ x \in \mathcal{V} \mid \langle m, x \rangle = 0 \quad \text{za } \forall m \in \mathcal{M} \}$$

Mi u stvari treba da odredimo koeficijente a, b, c i d takve da

$$\langle 1-x,\ a + bx + cx^2 + dx^3 \rangle = 0 \quad \text{i}$$

$$\langle x^2 - x,\ a + bx + cx^2 + dx^3 \rangle = 0$$

$$\int_0^1 (1-x)(a + bx + cx^2 + dx^3)\, dx = 0$$

$$\int_0^1 (x^2 - x)(a + bx + cx^2 + dx^3)\, dx = 0$$

za vježbu izračunati dva integrala ispod

Dobivamo da je

$$\frac{1\cdot5}{6\cdot5} - \frac{3\cdot3}{10\cdot3} = \frac{5-9}{30} = \frac{-4}{30}$$

$$\frac{a}{2} + \frac{b}{6} + \frac{c}{12} + \frac{d}{20} = 0 \quad /\cdot 2$$

$$-\frac{a}{6} - \frac{b}{12} - \frac{c}{20} - \frac{d}{30} = 0 \quad /\cdot 6$$

$$a + \frac{1}{3}b + \frac{1}{6}c + \frac{1}{10}d = 0$$

$$-a - \frac{1}{2}b - \frac{3}{10}c - \frac{1}{5}d = 0$$

$$\begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{6} & \frac{1}{10} \\ -1 & -\frac{1}{2} & -\frac{3}{10} & -\frac{1}{5} \end{bmatrix} \begin{matrix} II_r + I_r \\ \sim \end{matrix}$$

$$\sim \begin{bmatrix} 1 & 1/3 & 1/6 & 1/10 \\ 0 & -1/6 & -2/15 & -1/10 \end{bmatrix}$$

$\Rightarrow$ rang A = rang $\bar{A} = 3 < 4$

2 promjenjive uzimamo proizvoljno npr. $d = 10s$

$c = 15t$

$$a + \frac{1}{3}b + \frac{1}{6}c + \frac{1}{10}d = 0$$

$$-\frac{1}{6}b - \frac{2}{15}c - \frac{1}{10}d = 0 \quad \Rightarrow \quad -\frac{1}{6}b - 2t - s = 0$$

$$\frac{1}{6}b = -2t - s \quad /\cdot 6$$

$$b = -12t - 6s$$

$$a + \frac{1}{3}(-12t - 6s) + \frac{1}{6}\cdot 15t + s = 0$$

$$a \underset{\sim}{-4t} \underset{\approx}{-2s} + \underset{\sim}{\frac{5}{2}t} \underset{\approx}{+s} = 0$$

$$a = s + \frac{3}{2}t$$

$$\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} s + \frac{3}{2}t \\ -12t - 6s \\ 15t \\ 10s \end{pmatrix} = \begin{pmatrix} 3/2 \\ -12 \\ 15 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ -6 \\ 0 \\ 10 \end{pmatrix} s$$

Možemo zaključiti da je $\mathcal{L}^\perp = \text{span}\left\{\frac{3}{2} - 12x + 15x^2, 1 - 6x + 10x^3\right\}$

tj. baza za $\mathcal{L}^\perp$ je $\left\{\frac{3}{2} - 12x + 15x^2, 1 - 6x + 10x^3\right\}$.

Odredimo sad koeficijente α, β, γ i δ takve da

$$\alpha(1-x)+\beta(x^2-x)+\gamma\left(\frac{3}{2}-12x+15x^2\right)+\delta(1-6x+10x^3)=1-2x+5x^3$$

$$\begin{bmatrix} 1 & 0 & \frac{3}{2} & 1 \\ -1 & -1 & -12 & -6 \\ 0 & 1 & 15 & 0 \\ 0 & 0 & 0 & 10 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 0 \\ 5 \end{bmatrix}$$

$$\bar{A}=\left(\begin{array}{cccc|c} 1 & 0 & \frac{3}{2} & 1 & 1 \\ -1 & -1 & -12 & -6 & -2 \\ 0 & 1 & 15 & 0 & 0 \\ 0 & 0 & 0 & 10 & 5 \end{array}\right) \underset{IV_v : 5}{\overset{II_v + I_v}{\sim}} \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 1 \\ 0 & -1 & -21/2 & -5 & -1 \\ 0 & 1 & 15 & 0 & 0 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right) \overset{III_v + II_v}{\sim}$$

$$\sim \left(\begin{array}{cccc|c} 1 & 0 & 3/2 & 1 & 1 \\ 0 & -1 & -21/2 & -5 & -1 \\ 0 & 0 & 9/2 & -5 & -1 \\ 0 & 0 & 0 & 2 & 1 \end{array}\right) \cdots \overset{\text{za vježbu}}{\sim} \left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -5 \\ 0 & 0 & 1 & 0 & 1/3 \\ 0 & 0 & 0 & 1 & 1/2 \end{array}\right)$$

$$\Rightarrow \quad \alpha=0, \quad \beta=-5, \quad \gamma=1/3, \quad \delta=1/2$$

Prema tome

$$1-2x+5x^3=\underbrace{-5(x^2-x)}_{\in \mathcal{L}}+\underbrace{\frac{1}{3}\left(\frac{3}{2}-12x+15x^2\right)+\frac{1}{2}(1-6x+10x^3)}_{\in \mathcal{L}^\perp}$$

$$g(x)=-5x^2+5x \qquad r(x)=5x^3+5x^2-7x+1$$